

Gravity and the Quantum Vacuum Inertia Hypothesis

Alfonso Rueda

*Department of Electrical Engineering
California State University, 1250 Bellflower Blvd., Long Beach, CA 90840
arueda@csulb.edu*

Bernard Haisch

*Chief Science Officer
ManyOne Networks, 100 Enterprise Way, Bldg. G-370, Scotts Valley, CA 95066
haisch@calphysics.org*

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Abstract: In previous work it has been shown that the electromagnetic quantum vacuum, or electromagnetic zero-point field, makes a contribution to the inertial reaction force on an accelerated object. We show that the result for inertial mass can be extended to passive gravitational mass. As a consequence the weak equivalence principle, which equates inertial to passive gravitational mass, appears to be explainable. This in turn leads to a straightforward derivation of the classical Newtonian gravitational force. We call the inertia and gravitation connection with the vacuum fields the *quantum vacuum inertia hypothesis*. To date only the electromagnetic field has been considered. It remains to extend the hypothesis to the effects of the vacuum fields of the other interactions. We propose an idealized experiment involving a cavity resonator which, in principle, would test the hypothesis for the simple case in which only electromagnetic interactions are involved. This test also suggests a basis for the free parameter $\eta(\nu)$ which we have previously defined to parametrize the interaction between charge and the electromagnetic zero-point field contributing to the inertial mass of a particle or object.

Keywords: quantum vacuum, mass, zero-point field, inertia, gravitation, stochastic electrodynamics, principle of equivalence

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1. INTRODUCTION

It appears that the electromagnetic quantum vacuum should make a contribution to the inertial mass, m_i , of a material object in the sense that at least part of the inertial force of opposition to acceleration, or inertia reaction force, springs from the electromagnetic quantum vacuum [1,2,3]. The relevant properties of the electromagnetic quantum vacuum were calculated in a *Rindler frame* (a name we proposed for a rigid frame that performs uniformly accelerated motion) to find a force of opposition exerted by the quantum vacuum radiation to oppose the acceleration of an electromagnetically interacting object. We call this force associated with the quantum vacuum radiation in accelerating reference frames the *Rindler frame force*. It originates in an event horizon asymmetry for an accelerated reference frame. Owing to the symmetry and Lorentz invariance of quantum vacuum radiation in unaccelerated reference frames, the Rindler frame force is zero in constant velocity (inertial) frames. Not all radiation frequencies are equally effective in exerting this opposition. The effectiveness of the various frequencies may be parametrized by a function of the form $\eta(\omega)$ whose postulation is later justified by means of a compelling illustration. This suggests an experimental approach to measure the electromagnetic quantum vacuum contribution to inertial mass in a simple calculable situation. The motivation for our interpretation comes from the discovery that the resulting force proves to be proportional to acceleration, thus suggesting a basis for inertia of matter. [1,2,3] It thus appears that Newton's equation of motion could be *derived* in this fashion from electrodynamics, and it has been shown that the relativistic version of the equation of motion also naturally follows.

The energy of a quantum harmonic oscillator is allowed to take on only discrete values, $E_n = (n + \frac{1}{2})\hbar\omega$, implying a minimum energy of $\hbar\omega/2$. This can be viewed as a consequence of the Heisenberg uncertainty

principle. The radiation field is quantized by associating each mode with a harmonic oscillator [4]. This implies that there should exist a ground state of electromagnetic quantum vacuum, or zero-point, energy. Even though this appears to be an immediate and inescapable consequence of quantum theory, it is usually argued that such a field must be virtual, since the energy density of the field would be expected to have cosmological effects grossly inconsistent with the observed properties of the Universe, sometimes cited as a 120 order of magnitude discrepancy.

We set this objection temporarily aside to explore an intriguing connection between the properties of this radiation field and one of the fundamental properties of matter, i.e. mass. (This will, in fact, suggest an approach to resolving the discrepancy.) We do so using the techniques of stochastic electrodynamics (SED). Alternatively, we recently have used the techniques of quantum electrodynamics (QED) for obtaining exactly the same result [5]. Even this, though, may still be labelled a semiclassical result in the sense that we only consider the quantum structure of the field in its general form in the Rindler frame regardless of the details of the particles it may interact with.

If one assumes, as in SED, that the zero-point radiation field carries energy and momentum in the usual way, and if that radiation interacts with the particles comprising matter in the usual way, it can be shown that a law of inertia can be derived for matter comprised of electromagnetically interacting particles that are *a priori* devoid of any property of mass. In other words, the $\mathbf{f} = m\mathbf{a}$ law of mechanics — as well as its relativistic counterpart — can be traced back not to the existence in matter of mass (either innate or due to a Higgs mechanism), but to a purely electromagnetic effect (and possibly analogous contributions from other vacuum fields). It can be shown that the mass-like properties of matter reflect the energy-momentum inherent in the quantum vacuum radiation field. We call this the *quantum vacuum inertia hypothesis*.

There are additional consequences that make this approach of assuming real interactions between the electromagnetic quantum vacuum and matter appear promising. It can be shown that the weak principle of equivalence — the equality of inertial and gravitational mass ¹ — naturally follows. In the quantum vacuum inertia hypothesis, inertial and gravitational mass are not merely equal, they prove to be the identical thing. Inertial mass arises upon acceleration through the electromagnetic quantum vacuum, whereas gravitational mass — as manifest in weight — results from what may in a limited sense be viewed as acceleration of the electromagnetic quantum vacuum past a fixed object. The latter case occurs when an object is held fixed in a gravitational field and the quantum vacuum radiation associated with the freely-falling frame instantaneously comoving with the object follows curved geodesics as prescribed by general relativity.

Finally, the interactions of the quantum vacuum radiation field with massless particles results in Schrödinger's zitterbewegung, consisting of random speed-of-light fluctuations for a particle such as the electron. These fluctuations can be shown to cause a point-like particle to appear spread out in volume over a region — call it the Compton sphere — reminiscent of that predicted by the corresponding wave function. When this is viewed from a moving reference frame, the Doppler shift of these fluctuations results in an observed superimposed envelope that has properties like the de Broglie wave [7]. We thus tentatively suggest that it may be worth considering the so-called rest mass of particles to be a manifestation of the energy associated with zitterbewegung, especially since simulations are showing that in the presence of an external field, massless particles undergoing zitterbewegung will acquire helical, spin-like properties (L. J. Nickisch, priv. comm.).

It is thus suggestive that both mass and the wave nature of matter can be traced back to specific interactions with the electromagnetic zero-point field and possibly the other bosonic vacuum fields. Given this possible reinterpretation of fundamental properties, we suggest that it is premature to take a firm stand against the reality of the zero-point radiation field and its associated energy on the basis of cosmological arguments, especially given the possible relation between quantum vacuum, or zero-point, radiation and dark energy.

SED is a theory that includes the effects of the electromagnetic quantum vacuum in physics by adding to ordinary Lorentzian classical electrodynamics a random fluctuating electromagnetic background constrained

¹ When we write *gravitational mass* in this paper we mean *passive gravitational mass* unless otherwise specified. Only for point particles or for spherically symmetric stationary matter distributions can passive and active gravitational mass densities be deemed identical. In the general case they are not the same. This difference mainly appears in cosmology, where passive and active gravitational mass densities in general differ [6].

to be homogeneous and isotropic and to look exactly the same in every Lorentz inertial frame of reference [8,9,10]. This replaces the zero homogeneous background of ordinary classical electrodynamics. It is essential that this background not change the laws of physics when exchanging one inertial reference system for another. This translates into the requirement that this random electromagnetic background must have a Lorentz invariant energy density spectrum. The only random electromagnetic background with this property is one whose spectral energy density, $\rho(\omega)$, is proportional to the cube of the frequency, $\rho(\omega)d\omega \sim \omega^3 d\omega$. This is the case if the energy per mode is $\hbar\omega/2$ where ω is the angular frequency, the other factor of ω^2 coming from the density of modes [4]. (The $\hbar\omega/2$ energy per mode is of course also the minimum energy of the analog of an electromagnetic field mode: a harmonic oscillator.) The spectral energy density required for Lorentz invariance is thus identical to the spectral energy density of the zero-point field of ordinary quantum theory. For most purposes, including the present one, the zero-point field of SED may be identified with the electromagnetic quantum vacuum [10]. However SED is essentially a classical theory since it presupposes only ordinary classical electrodynamics and hence SED also presupposes special relativity (SR). Our reliance on SED [1,2,3] has been due to the fact that SED allows a very clear intuitive representation. Nevertheless, the SED-based results of [1,2] have been readily rederived within a QED formalism [5].

In §2 we briefly review a previously-derived development of the quantum vacuum inertia hypothesis. In §3 we introduce an example that allows a very clear visualization of the coupling of an electromagnetically interacting object with the surrounding vacuum fields and the evaluation of the function $\eta(\omega)$ mentioned above [1,2]. We turn next to the principle of equivalence. According to the weak equivalence principle (WEP) of Newton and Galileo, inertial mass is equal to gravitational mass, $m_i = m_g$. If the quantum vacuum inertia hypothesis is correct, a very similar mechanism involving the quantum vacuum should also account for gravitational mass. This novel result, restricted for the time being to the electromagnetic vacuum component, is precisely what we show in §4 by means of formal but simple and straightforward arguments requiring physical assumptions that are uncontroversial and widely accepted in theoretical physics. In §5 the consistency of this argument with so-called metric theories of gravity (i.e. those theories characterized by spacetime curvature) is exhibited. In addition to the metric theory, *par excellence*, Einstein's GR, there is the Brans-Dicke theory and other less well known ones, briefly discussed by Will [11]. A non-spacetime curvature theory is briefly discussed in §6.

Nothing in our approach points to any new discriminants among the various metric theories. Nevertheless, our quantum vacuum approach to gravitational mass will be shown to be entirely consistent with the standard version of GR. Next, in §7, we take advantage of geometrical symmetries and present a short argument supported by standard potential theory to show that in the weak field limit a Newtonian inverse square force must result.

A new perspective on the origin of weight is presented in §8. In §9 we discuss an energetics aspect, related to the derivation presented herein, and resolve an apparent paradox. A brief discussion on the nature of the gravitational field follows in §10, but we infer that the present development of our approach does not provide any deeper or more fundamental insight than GR itself into the ability of matter to bend spacetime. We present conclusions in §11.

2. THE RINDLER FRAME FORCE AND INERTIA

It has been shown [1,2,3] that the electromagnetic quantum vacuum makes a contribution to the inertial mass, m_i , of a material object in the sense that at least part of the inertial force of opposition to acceleration, or inertia reaction force, springs from the electromagnetic quantum vacuum. Specifically, in the previously cited work, the properties of the electromagnetic quantum vacuum as experienced in a Rindler, or constant proper acceleration, frame were investigated, and we calculated a resulting force of opposition on material objects fixed in such a Rindler non-inertial frame and interacting with the electromagnetic zero-point radiation field. It originates in an event horizon asymmetry for an accelerated reference frame. Owing to the symmetry and Lorentz invariance of zero-point radiation in unaccelerated reference frames, the Rindler frame force vanishes at constant velocity, i.e. in inertial frames. We further hypothesized that if V_0 is the proper volume of the material object, some frequency-dependent fraction, $\eta(\omega)$, of this radiative momentum will, via interactions, be conferred upon the particles comprising the matter. The motivation for this interpretation comes from the discovery that the resulting Rindler frame force proves to be proportional to acceleration. It thus appears that Newton's equation of motion can be *derived* in this fashion from electrodynamics, and it has been shown

that the relativistic version of the equation of motion also naturally follows. We have called the notion that at least part of the inertia of an object should be due to the individual and collective interaction of its quarks and electrons with the quantum vacuum as the *quantum vacuum inertia hypothesis* with the proviso that analogous contributions are expected from the other bosonic vacuum fields.

The electromagnetic field inside a cavity of perfectly reflecting walls can be shown to be expandable in countably infinite different modes where each mode corresponds to an independent oscillation which always, even in the case of free space, can be represented as a harmonic oscillator [12]. When quantized, the harmonic oscillator gives an energy of the form $\hbar\omega(n + 1/2)$, where ω is the characteristic frequency of the mode ($\omega = 2\pi\nu$), and when the integer n is set to zero, there is still a remnant energy $\hbar\omega/2$. Alternatively we also have that for the harmonic oscillator, the Heisenberg uncertainty relation tells us that the minimum (zero-point) energy is $\hbar\omega/2$. The electromagnetic field therefore has a minimum quantum energy state consisting of zero-point fluctuations having an average energy per mode of $\hbar\omega/2$. The density of modes is, see e.g. [4],

$$N(\omega)d\omega = \frac{\omega^2 d\omega}{\pi^2 c^3}, \quad (1)$$

and this combined with the average energy per mode of $\hbar\omega/2$ yields a spectral energy density for the zero-point fluctuations of

$$\rho(\omega)d\omega = \frac{\hbar\omega^3}{2\pi^2 c^3} d\omega. \quad (2)$$

In the semi-classical approach known as stochastic electrodynamics (SED) the quantum fluctuations of the electric and magnetic fields are treated as random plane waves summed over all possible modes with each mode having this $\hbar\omega/2$ energy. The electric and magnetic zero-point fluctuations in the SED approximation are thus

$$\mathbf{E}^{zp}(\mathbf{r}, t) = \sum_{\lambda=1,2} \int d^3k \left(\frac{\hbar\omega}{2\pi^2} \right)^{1/2} \hat{\epsilon}(\mathbf{k}, \lambda) \cos[\mathbf{k} \cdot \mathbf{r} - \omega t - \theta(\mathbf{k}, \lambda)], \quad (3a)$$

$$\mathbf{B}^{zp}(\mathbf{r}, t) = \sum_{\lambda=1,2} \int d^3k \left(\frac{\hbar\omega}{2\pi^2} \right)^{1/2} [\hat{k} \times \hat{\epsilon}(\mathbf{k}, \lambda)] \cos[\mathbf{k} \cdot \mathbf{r} - \omega t - \theta(\mathbf{k}, \lambda)], \quad (3a)$$

where the sum is over the two polarization states, $\hat{\epsilon}$ is a unit vector, and $\theta(\mathbf{k}, \lambda)$ is a random variable uniformly distributed in the interval $(0, 2\pi)$. For simplicity and to facilitate comparison with much previous SED work, we omit the Ibison and Haisch modification of eqns. (3) in which the amplitudes are also randomized in such a way as to bring the quantum and classical statistics of the electromagnetic zero-point field into exact agreement. [13]

Since the \mathbf{E}^{zp} and \mathbf{B}^{zp} field fluctuations are entirely random, there is no net energy-momentum flow across any surface, or in other words, the value of the stochastically averaged Poynting vector must be zero:

$$\langle \mathbf{N}^{zp} \rangle = \frac{c}{4\pi} \langle \mathbf{E}^{zp} \times \mathbf{B}^{zp} \rangle = 0. \quad (4)$$

It is nowadays well known how to transform the \mathbf{E}^{zp} and \mathbf{B}^{zp} field vectors from a stationary frame to one undergoing uniform acceleration, i.e. having constant proper acceleration, that we have labelled a Rindler frame. Such a frame will experience an asymmetric event horizon leading to a non-zero electromagnetic energy and momentum flux as calculated by a stationary observer. The velocity and the Lorentz factor for such a frame are,

$$\frac{v}{c} = \tanh\left(\frac{\alpha\tau}{c}\right), \quad (5a)$$

$$\gamma_\tau = \cosh\left(\frac{\alpha\tau}{c}\right), \quad (5b)$$

where α is the object's proper acceleration and τ its proper time.

The general, compact form of the Lorentz transformation of electromagnetic fields is (cf. eqn. 11.149 in [14])

$$\mathbf{E}' = \gamma(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}) - \left(\frac{\gamma^2}{\gamma+1}\right) \frac{\mathbf{v}}{c} \left(\frac{\mathbf{v}}{c} \cdot \mathbf{E}\right), \quad (6a)$$

$$\mathbf{B}' = \gamma(\mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E}) - \left(\frac{\gamma^2}{\gamma+1}\right) \frac{\mathbf{v}}{c} \left(\frac{\mathbf{v}}{c} \cdot \mathbf{B}\right). \quad (6b)$$

Particularizing \mathbf{v} to be along the x -direction and using component form of eqns. (6), one transforms \mathbf{E}^{zp} and \mathbf{B}^{zp} to the Rindler frame (in a way explained in considerably more detail in [1]) in order to obtain:

$$\begin{aligned} \mathbf{E}^{zp}(0, \tau) = \sum_{\lambda=1,2} \int d^3k \left(\frac{\hbar\omega}{2\pi^2}\right)^{1/2} \left\{ \hat{x}\hat{\epsilon}_x + \hat{y} \cosh\left(\frac{\alpha\tau}{c}\right) \left[\hat{\epsilon}_y - \tanh\left(\frac{\alpha\tau}{c}\right) (\hat{k} \times \hat{\epsilon})_z\right] \right. \\ \left. + \hat{z} \cosh\left(\frac{\alpha\tau}{c}\right) \left[\hat{\epsilon}_z + \tanh\left(\frac{\alpha\tau}{c}\right) (\hat{k} \times \hat{\epsilon})_y\right] \right\} \cos \left[k_x \frac{c^2}{a} \cosh\left(\frac{\alpha\tau}{c}\right) - \frac{\omega c}{a} \sinh\left(\frac{\alpha\tau}{c}\right) - \theta(\mathbf{k}, \lambda) \right], \quad (7a) \end{aligned}$$

$$\begin{aligned} \mathbf{B}^{zp}(0, \tau) = \sum_{\lambda=1,2} \int d^3k \left(\frac{\hbar\omega}{2\pi^2}\right)^{1/2} \left\{ \hat{x}(\hat{k} \times \hat{\epsilon})_x + \hat{y} \cosh\left(\frac{\alpha\tau}{c}\right) \left[(\hat{k} \times \hat{\epsilon})_y + \tanh\left(\frac{\alpha\tau}{c}\right) \hat{\epsilon}_z\right] \right. \\ \left. + \hat{z} \cosh\left(\frac{\alpha\tau}{c}\right) \left[(\hat{k} \times \hat{\epsilon})_z - \tanh\left(\frac{\alpha\tau}{c}\right) \hat{\epsilon}_y\right] \right\} \cos \left[k_x \frac{c^2}{a} \cosh\left(\frac{\alpha\tau}{c}\right) - \frac{\omega c}{a} \sinh\left(\frac{\alpha\tau}{c}\right) - \theta(\mathbf{k}, \lambda) \right], \quad (7b) \end{aligned}$$

where $\hat{\epsilon}_x$ signifies the scalar projection of the $\hat{\epsilon}$ unit vector along the x -direction, and similarly for $\hat{\epsilon}_y$ and $\hat{\epsilon}_z$.

Our goal here has been to calculate the rate of change of the momentum applied by the zero-point field on the electromagnetically interacting, accelerating object. Each individual inertial frame has associated with itself its own (or proper) random electromagnetic zero-point field background and thus its proper zero-point field vacuum spectral energy density distribution. For each inertial frame its proper electromagnetic vacuum is homogeneously and isotropically distributed. This means that the components of the net Poynting vector of the zero-point field of a given frame when observed in that same frame all should vanish. Moreover, of the components of the 4×4 electromagnetic energy-momentum stress tensor only the diagonal components survive as all other components should be zero. Extreme care should be exercised when performing such Lorentz transformations. Each of the components of the tensor is represented by an improper integral over k -space, which is the k -space representation associated with the particular inertial frame that we are dealing with. When transforming to another inertial frame it is not enough to transform the interior of the integral, i.e. its integrand. One must also take care of what in Appendix C of [1] we called the k -sphere. This is done by means of a cutoff convergence factor e^{-k/k_c} that effectively cuts off the zero-point radiation beyond some maximal magnitude of the wave vector in the k -space of the original inertial frame, thus the name k -sphere. The cutoff parameter k_c clearly makes the zero-point energy as well as all other components of the tensor, like the Poynting vector components and the Maxwell 3×3 stress tensor components, all finite, which is a desirable consequence. Nevertheless, the spectral Lorentz invariance holds even when at the end we make $k_c \rightarrow \infty$ anyway. In such cases e^{-k/k_c} becomes just a form factor to regularize the integration [1].

Observe however that in the many integrations we performed in [1] we really did not need to use such a cutoff factor. The reason is clear: the final integrations were performed at accelerating object proper time $\tau = 0$, when the laboratory frame and the accelerated-particle frame instantaneously comoved. Indeed, we observe that all final integrations for the inertia reaction force in [1] are performed at $\tau = 0$, which is the time instant when the uniformly-accelerated object comoves with the inertial laboratory frame. This means that both the momentum density, and thereby the Poynting vector, of the zero-point field are for the object, those of the laboratory frame.

But there is a more subtle change that needs to be captured. Both the momentum density and the total momentum of the zero-point field inside the volume V_0 of the body display a time rate of change and have a non-vanishing time derivative. Thus although both the Poynting vector and the momentum of the zero-point field inside the body instantaneously vanish, their time derivatives at that coincidence time do

not vanish. This can also be visualized as a manifestation of a reduction in symmetry of the event horizon that passes from the perfect three-dimensional symmetry of a sphere to a lower two-dimensional symmetry that is merely axial. This results in a special situation that may be visualized as a stress or tension in the vacuum field which is manifested on the accelerated particle as the Rindler frame force.

From the electromagnetic zero-point field in the Rindler frame of the accelerated object we calculate the Poynting vector (see [1] for details). For uniform acceleration in the x -direction, the Poynting vector is

$$\mathbf{N}^{zp}(\tau) = -\hat{x} \frac{2c}{3} \sinh\left(\frac{2\alpha\tau}{c}\right) \int \frac{\hbar\omega^3}{2\pi^2 c^3} d\omega = -\hat{x} \frac{2c}{3} \sinh\left(\frac{2\alpha\tau}{c}\right) \int \rho(\omega) d\omega, \quad (8)$$

where $\rho(\omega)$ is the spectral energy density of the zero-point fluctuations.

The amount of radiative momentum carried by the zero-point fluctuations that are passing through and instantaneously contained in the accelerated object which is undergoing uniform proper acceleration $\vec{\alpha} = \alpha\hat{x}$ and that has proper volume V_0 , is

$$\mathbf{p}^{zp} = -\hat{x} \frac{V_0}{\gamma_\tau} \frac{2}{3c} \sinh\left(\frac{2\alpha\tau}{c}\right) \int \rho(\omega) d\omega = -\hat{x} \frac{4}{3} V_0 v_\tau \gamma_\tau \left[\frac{1}{c^2} \int \rho(\omega) d\omega \right]. \quad (9)$$

Let us assume that at frequency ω some small fraction $\eta(\omega)$ of this energy interacts with the particles of matter contained in V_0 . We can then write the interacting fraction of the zero-point momentum instantaneously contained in and passing through the object as

$$\mathbf{p}^{zp} = -\hat{x} \frac{4}{3} V_0 v_\tau \gamma_\tau \left[\frac{1}{c^2} \int \eta(\omega) \rho(\omega) d\omega \right]. \quad (10)$$

If we take the time derivative of this quantity, we find that [1]

$$\frac{d\mathbf{p}^{zp}}{dt} = \mathbf{f}^{zp} = - \left[\frac{4}{3} \frac{V_0}{c^2} \int \eta(\omega) \rho(\omega) d\omega \right] \mathbf{a}. \quad (11)$$

A fully covariant analysis eliminates the factor of $4/3$ and also yields a proper relativistic four-vector force expression (see Appendix D of [1]). Eqn. (11) is telling us that in order to maintain the acceleration of such an object, a motive force, \mathbf{f} , must continuously be applied to balance the vacuum electromagnetic counteracting reaction force, that we may call the Rindler frame force, \mathbf{f}^{zp} . The motive force is,

$$\mathbf{f} = -\mathbf{f}^{zp} = \left[\frac{V_0}{c^2} \int \eta(\omega) \rho(\omega) d\omega \right] \mathbf{a}. \quad (12)$$

where we have suppressed the factor of $4/3$. This strongly suggests Newton's $\mathbf{f} = m\mathbf{a}$, and that the zero-point field contribution to the inertial mass contained in V_0 is the mass-like coefficient

$$m_i = \left[\frac{V_0}{c^2} \int \eta(\omega) \rho(\omega) d\omega \right]. \quad (13)$$

In other words, some of the apparent inertial mass of an object originates in the interacting fraction of the zero-point energy instantaneously contained in an object. In this view, the apparent momentum of the object can be traced back to the momentum of the zero-point radiation field.

We suspect that $\eta(\omega)$ involves some kind of resonance at the Compton frequency of individual particles, ($\omega_C = mc^2/\hbar$), since this suggests a close connection between the origin of mass and the de Broglie wavelength, both stemming from interactions of matter with the quantum vacuum [7].

We speculatively digress for a moment in order to try to gain some physical insight into how the interrelationships of minutely different coordinate systems can translate into real measurable effects. Since momentum is not absolute but relative to the observer, and indeed always zero in one's own frame, one wonders how the time derivative of this arbitrary quantity can yield something real and absolute, i.e. a force. We suggest that the answer is deeply rooted in the basis of special relativity that all interactions are limited by the speed of light. Hence if one pushes on an object at point A, it will take a finite time for that push to be transferred to point B in the same object. The acceleration of any object of finite extent involves a co-mingling of an infinite number of minutely different reference frames. Acceleration necessarily

mixes infinitely many adjacent reference frames as a result of the propagation limitations of special relativity. This analogously suggest a physical meaning for the otherwise seemingly purely mathematical operation of calculating how \mathbf{E}^{zp} and \mathbf{B}^{zp} would look to an observer in one (inertial) reference frame from the basis of another (accelerating).

In Appendix B of [1] we also showed an alternative but complementary approach. Instead of considering the opposition to acceleration due to the electromagnetic zero-point field background through which the object is being accelerated (of the previous approach in the body of Ref. [1]) we also looked at the amount of electromagnetic energy contained within the object according to the viewpoint of an inertial stationary observer (Appendix B of [1]). The very natural result that ensues for that stationary observer is that as the object moves faster and faster the amount of enclosed energy of zero-point radiation grows in exact proportion to γ_τ , which is exactly the same way as the mass of a moving and accelerating object of rest mass m_i behaves according to an inertial stationary observer. In this complementary approach the rest mass turns out to be exactly the same as in the zero-point field drag approach of eqn. (13) above. Moreover, whereas in eqn. (10) for the zero-point field drag approach we obtained the force \mathbf{f}^{zp} that the zero-point field exerts on the object when opposing its acceleration, in the new approach we directly get \mathbf{f} , the force that the external agent has to apply to the object to accelerate it, which we already noticed in eqn. (12).

The formal development, i.e. the equations in the derivation, of this complementary approach comes out to be superficially similar to the formal development of the zero-point field drag approach, except that all equations since the very first ones [1] have the opposite sign than in the previous case. Conceptually though, the two approaches are not similar. Instead they are very complementary. We could say that they are like the two sides of the same coin. One cannot exist without the other, but they are distinctly different.

The rest mass in eqn. (13) exactly corresponds to the amount of zero-point field mass equivalent enclosed within the volume of the body and that, thanks to the $\eta(\omega)$ spectral factor interacts with the body. If the mass is moving with respect to a stationary observer, with speed v_τ , the associated momentum is \mathbf{p} and in the second approach comes as given by eqn. (11) but with the positive sign. (Recall that the 4/3 factor comes from not using a covariant approach. The covariant approach of Appendix D [1] removes the spurious factor by considering all components in the electromagnetic stress-energy momentum tensor.)

This is analogous to the situation of relativistic mass increase. A stationary observer would come to the conclusion that the mass of an accelerating object is steadily increasing as $\gamma_\tau m$, but in the accelerating frame no mass change is evident. One's momentum in one's own reference frame is always zero. Physical consequences only ensue when there is a change in momentum, akin to a "physical" change from one reference frame into another.

In the approach of the quantum vacuum inertia hypothesis, it becomes physically evident how the Lorentz factor γ_τ , which characterizes a space-time geometry relationship, acquires the physical role of relativistic mass increase parameter. *It also becomes clear why the relativistic mass increase must become infinite at c: One cannot propagate the effect of any forces due to the zero-point field at speeds faster than c.*

3. ON THE ELECTROMAGNETIC MODEL OF THE ACCELERATED OBJECT

The second or complementary approach of [1], Appendix B, that we mentioned above, very clearly shows that $m_i c^2$ is just the amount of zero-point energy located inside the accelerated object and that instantaneously comoves with the object as viewed by an inertial observer.

Here we would like to visualize the meaning of the parameter V_0 , the volume of the object in an electromagnetic sense, and the "efficiency" or "interaction" function $\eta(\omega)$. For this we just need to think of an electromagnetic cavity comprised of surrounding conducting walls. In such a case V_0 is just the internal volume of the cavity. The most familiar shape is a rectangular parallelepiped of sides a , b and c , with $a < b < c$. We consider here only the volume inside the cavity and as far as m_i goes, the mass is just $1/c^2$ times the total amount of zero-point field energy that can be accumulated within the structure of the cavity.

The cavity walls are conducting and as such serve to confine the electromagnetic oscillating radiation within its boundaries. Internally, the cavity can sustain a countable and finite number of characteristic or proper modes of oscillation. It is well known that beyond a threshold frequency ω_p that corresponds to the plasma frequency of the electrons in the conducting walls, microwave cavities become essentially transparent and cannot sustain any more modes internally. The lowest frequency modes are comparatively widely distributed in frequency but as the characteristic frequencies of the modes increase, they become

more closely spaced in frequency. For wavelengths λ of the modes much smaller than the dimensions of the cavity, $\lambda \ll a < b < c$, their frequency number density starts to grow roughly in proportion to $\omega^2/\pi^2 c^3$ which, not surprisingly, is the $N(\omega)$ of eqn. (1). Recall that $N(\omega)$ is obtained when the cavity that encloses the radiation grows in size by letting $c > b > a \rightarrow \infty$, which yields what we call the *limit of the continuum*.

The walls, because they are conducting, serve to indirectly connect the inside of the cavity to the outside. At zero temperature, on the outside we have the zero-point electromagnetic radiation of eqns. (1) and (2) and in the inside only the frequencies corresponding to the modes which can be excited. The modes of the cavity weakly couple through the conducting walls to the random electromagnetic radiation outside. At zero temperature, the cavity l^{th} mode, being a harmonic oscillator, acquires the oscillation of energy $\hbar\omega_l/2$ where ω_l represents the l^{th} mode characteristic frequency.

Let us first assume that the cavity is an ideal one in which dissipation may be neglected. This by itself does not preclude some broadening around the characteristic frequency because there are, in the walls, quantum fluctuations of the electrons and of the plasma of electrons at zero temperature which induce some small albeit nonvanishing blurring of the exact mode characteristic frequency due to the Doppler shifts and fluctuations in the cavity geometry (dimensions a , b and c are not exact anymore). Hence there is always some broadening.

For the time being, however, let us neglect such broadening and consider instead an ideal cavity with a finite number of discrete modes up to ω_p , the plasma frequency of the electrons in the walls, and at strictly zero temperature, $T = 0$. Each mode then oscillates at its exact characteristic frequency, ω_k , and with an energy corresponding to that of the harmonic oscillator zero-point oscillation, $\hbar\omega_k/2$, $k = 1, 2, \dots, N$, where N is the maximal mode frequency such that $\omega_N \leq \omega_p$. The total energy is then

$$E = \sum_{k=1}^N \frac{\hbar\omega_k}{2}, \quad \omega_1 \leq \omega_2 \leq \omega_3 \leq \dots \omega_N \leq \omega_p. \quad (14)$$

But this energy must be exactly equal to the energy given by eqn. (13):

$$m_i c^2 = \int_{\omega=0}^{\infty} V_0 \eta(\omega) \rho(\omega) d\omega = E = \sum_{k=1}^N \frac{\hbar\omega_k}{2}. \quad (15)$$

As $\rho(\omega)$ is given by eqn. (2), it is easy to see that in order to satisfy (15) we need an $\eta(\omega)$ of the form:

$$\eta(\omega) = \sum_{k=1}^N \frac{1}{V_0} \cdot \frac{\pi^2 c^3}{\omega^2} \delta(\omega - \omega_k) = \sum_{k=1}^N \frac{\pi^2 c^3}{V_0} \frac{1}{\omega^2} \delta(\omega - \omega_k). \quad (16)$$

The idealized cavity that we have considered gives rise to an $\eta(\omega)$ that essentially consists of a finite sum of idealized line-shaped functions of the usual form present when dissipation is neglected.

A better model is obtained when we allow for some dissipation in the interaction of the modes with the walls, or if no dissipation is considered, some broadening must still be present because of the reasons indicated above. It is well-known that in many cases, if the dissipation is small (see, e.g. [15,16,17]), that the lineshape function is the so-called Lorentzian lineshape function

$$g(\omega) d\omega = \frac{\Delta\omega}{2\pi \left[(\omega - \omega_0)^2 + \left(\frac{\Delta\omega}{2} \right)^2 \right]} d\omega. \quad (17)$$

The ubiquity of the Lorentzian lineshape comes from the fact that it originates in a dissipative exponential time-dependent decay (see [18]). The function $g(\omega)$ is called the Lorentzian lineshape function and it is such that

$$\int_0^{\infty} g(\omega) d\omega = 1 \quad (18)$$

and that

$$\lim_{\Delta\omega \rightarrow 0} g(\omega) = \delta(\omega - \omega_0). \quad (19)$$

The lineshape broadening parameter $\Delta\omega$ is due to various forms of dissipation and other broadening effects — such as those mentioned above — and $4\pi/\Delta\omega$ is the exponential time constant of decay for the relevant dissipative part of the process.

In the more realistic case when line broadening around the modes characteristic frequencies is considered, combining the inputs from (16) and (17) yields

$$\mathcal{K}(\omega) \equiv V_0\eta(\omega) = \frac{\pi^2 c^3}{\omega^2} \sum_{k=1}^N \frac{\Delta\omega_k}{2\pi \left[(\omega - \omega_k)^2 + \left(\frac{\Delta\omega_k}{2} \right)^2 \right]} \quad (20)$$

where, as explained above, the ω_k is the characteristic frequency of the k^{th} mode and $\Delta\omega_k$, with $\Delta\omega_k > 0$, its corresponding line broadening. Only at the highest frequencies in the microwave cavity are the modes expected to overlap because the $\Delta\omega_k$ broadenings become comparable to the frequency separations between the modes.

All in all the corresponding contribution of eqn. (20) to the inertial mass of a physical cavity resonator when the walls are included is fairly small, but what is in principle interesting is that it is not negligible! The inertial mass associated with the cavity itself (without the mass of the walls) of eqn. (20) becomes

$$\begin{aligned} m_i &= \frac{1}{c^2} \int_{\omega=0}^{\infty} V_0 \eta(\omega) \rho(\omega) d\omega \\ &= \int_{\omega=0}^{\infty} \frac{\pi^2 c}{\omega^2} \sum_{k=1}^N \frac{\Delta\omega_k}{2\pi \left[(\omega - \omega_k)^2 + \left(\frac{\Delta\omega_k}{2} \right)^2 \right]} \cdot \frac{\hbar\omega^3}{2\pi^2 c^3} d\omega \\ &= \frac{1}{c^2} \sum_{k=1}^N \int_{\omega=0}^{\infty} \frac{\Delta\omega_k}{2\pi \left[(\omega - \omega_k)^2 + \left(\frac{\Delta\omega_k}{2} \right)^2 \right]} \cdot \frac{\hbar\omega}{2} d\omega \end{aligned} \quad (21)$$

which is a rather interesting expression. We clearly just add up the zero-point energies of all the resonant cavity modes broadened by their corresponding Lorentzian broadening factors.

Observe that in eqn. (21) the volume V_0 of the cavity seemingly has disappeared. Of course that is just a superficial remark as the volume as well as the geometry are determining factors in the spectrum of frequencies $\{\omega_k\}$ distribution up to the plasma frequency of the cavity, ω_p , with $\omega_N \leq \omega_p$. The advantage for the general case of a more general object (or particle) is that V_0 is a parameter that in eqn. (21) has disappeared and been replaced by electromagnetic quantities like the ω_k and $\Delta\omega_k$ that, among other things, depend on the geometry.

In the case of a microwave cavity resonator mode with good conducting walls, relatively, the decay times of the modes are very long and consequently the $\Delta\omega_k$ parameters are very small with $\Delta\omega_k \ll \omega_k$. This allows us to propose a simple experiment to measure the electromagnetic quantum vacuum contribution to inertial mass.

If the $\Delta\omega_k$ are indeed negligible for a given conductor for all modes ω_k up to the maximal one ω_N , where $\omega_N \leq \omega_p$, and where ω_p is the plasma frequency of the conduction-band electrons in the metal of the walls, then the contribution to inertial mass given by the cavity is as given by eqn. (15), a quantity which in principle is readily calculable. This is particularly so for a cavity of rectangular shape with noncommensurable $a < b < c$ in order to avoid mode degeneracies.

Observe that the walls are made of a conductor and their crystal structure. In turn we expect that several vacuum fields beside the electromagnetic are going to make contributions to the mass of the walls, but in the cavity by itself the only contribution can be that of eqn. (15) which is purely electromagnetic. This suggests a simple experimental procedure in principle.

First, weigh in a precision balance the cavity with its walls in cryogenic conditions, i.e. at T as low as possible. Then disassemble the cavity and weigh its components. The first measurement under careful experimental conditions should yield a slightly larger mass than the second one. This excess mass must be the one associated with the energy E in eqn. (15). This is of course an extremely idealized experiment.

The experiment, if properly accomplished, would give interesting confirmations. First, some confirmation would be given to the electromagnetic vacuum contribution to inertial mass concept, or quantum vacuum inertia hypothesis. An important confirmation would be the reality — or virtuality — of the zero-point field. The reality of the zero-point field or quantum vacuum would be unquestionably established. The difficulty in the experiment is that even though there are many modes inside a cavity resonator, the mass associated with eqn. (15) should be very small even for a large cavity.

4. THE PHYSICAL BASIS OF THE PRINCIPLE OF EQUIVALENCE

We intend to show not only that the quantum vacuum inertia hypothesis is consistent with GR, but that it answers an outstanding question regarding a possible physical origin of the force manifesting as weight [19]. We also intend to show that just as it becomes possible to identify a physical process underlying the $\mathbf{f}=\mathbf{m}\mathbf{a}$ postulate of Newtonian mechanics (as well as its extension to SR [1]), it is possible to identify a parallel physical process underlying the weak equivalence principle, $m_i = m_g$.

Within the standard theoretical framework of GR and related theories, the equality (or proportionality) of inertial mass to gravitational mass has to be assumed. It remains unexplained. As correctly stated by Rindler [20], “the proportionality of inertial and gravitational mass for different materials is really a very mysterious fact.” However here we show that — at least within the present restriction to electromagnetism — the quantum vacuum inertia hypothesis leads naturally and inevitably to this equality. The interaction between the electromagnetic quantum vacuum and the electromagnetically-interacting particles constituting any physical object (quarks and electrons) is identical for the two situations of (a) acceleration with respect to constant velocity inertial frames or (b) remaining fixed above some gravitating body with respect to freely-falling local inertial frames.

A related theoretical lacuna involves the origin of the force which manifests itself as weight. Within GR theory one can only state that deviation from geodesic motion results in a force which must be an inertia reaction force. We propose that it is possible in principle to identify a mechanism which generates such an inertia reaction force, and that in curved spacetime it acts in the same way as acceleration does in flat spacetime.

Begin by considering a macroscopic, massive, gravitating object, W , which is fixed in space and which for simplicity we assume to be solid, of constant density, and spherical with a radius R , e.g. a planet-like object. At a distance $r \gg R$ from the center of W there is a small object, w , that for our purposes we may regard as a point-like test particle. A constant force \mathbf{f} is exerted by an external agent that prevents the small body w from falling into the gravitational potential of W and thereby maintains w at a fixed point in space above the surface of W . Experience tells us that when the force \mathbf{f} is removed, w will instantaneously start to move toward W with an acceleration \mathbf{g} and then continue freely falling toward W .

Next we consider a freely falling *local* inertial frame I_* (in the customary sense given to such a local frame [21]) that is instantaneously at rest with respect to w . At w proper time τ , that we select to be $\tau = 0$, object w is instantaneously at rest at the point $(c^2/g, 0, 0)$ of the I_* frame. The x -axis of that frame goes in the direction from W to w and, since the frame is freely falling toward W , at $\tau = 0$ object w appears accelerated in I_* in the x -direction and with an acceleration $\mathbf{g}_w = \hat{x}g$. As argued below, w is performing a uniformly-accelerated motion, i.e. a motion with a constant proper acceleration \mathbf{g}_w as observed from any neighboring instantaneously comoving (local) freely-falling inertial frame. In this respect we introduce an infinite collection of local inertial frames I_τ , with axes parallel to those of I_* and with a common x -axis which is that of I_* . Let w be instantaneously at rest and co-moving with the frame I_τ at w proper time τ . So the τ parameter representing the w proper time also serves to parametrize this infinite collection of (local) inertial frames. Clearly then, I_* is the member of the collection with $\tau = 0$, so that $I_* = I_{\tau=0}$. At the point in time of coincidence with a given I_τ , w is found momentarily at rest at the $(c^2/g, 0, 0)$ point of the I_τ frame. We select also the times in the (local) inertial frames to be t_τ and such that $t_\tau = 0$ at the moment of coincidence when w is instantaneously at rest in I_τ and at the aforementioned $(c^2/g, 0, 0)$ point of I_τ . Clearly as $I_{\tau=0} = I_*$ then $t_* = 0$ when $\tau = 0$. All the frames in the collection are freely falling toward W , and when any one of them is instantaneously at rest with w it is instantaneously falling with acceleration $\mathbf{g} = -\mathbf{g}_w = -\hat{x}g$ with respect to w in the direction of W . It is not difficult to realize that w appears in those frames as uniformly accelerated and hence performing hyperbolic motion with constant proper acceleration \mathbf{g}_w .

This situation is equivalent to that of an object w accelerating with respect to an ensemble of I_τ reference frames in the absence of gravity. In that situation, the concept of the ensemble of inertial frames, I_τ , each with an infinitesimally greater velocity (for the case of positive acceleration) than the last, and each coinciding instantaneously with an accelerating w is not difficult to picture. But how does one picture the analogous ensemble for w held stationary with respect to a gravitating body?

We are free to bring reference frames into existence at will. Imagine bringing a reference frame into existence at time $\tau = 0$ directly adjacent to w , but whereas w is fixed at a specific point above W , we let the newly created reference frame immediately begin free-falling toward W . We immediately create a replacement reference frame directly adjacent to w and let it drop, and so on. The ensemble of freely-falling local inertial frames bear the same relation to w and to each other as do the extended I_τ inertial frames used in the case of true acceleration of w [1,2].

For convenience we introduce a special frame of reference, S , whose x -axis coincides with those of the I_τ frames, including of course I_* , and whose y -axis and z -axis are parallel to those of I_τ and I_* . This frame S stays collocated with w which is positioned at the $(c^2/g, 0, 0)$ point of the S frame. For I_* (and for the I_τ frames) the frame S appears as accelerated with the uniform acceleration \mathbf{g}_w of its point $(c^2/g, 0, 0)$. We will assume that the frame S is rigid. If so, the accelerations of points of S sufficiently separated from the w point $(c^2/g, 0, 0)$ are not going to be the same as that of $(c^2/g, 0, 0)$. This is not a concern however since we will only need in all frames (I_τ , I_* and S) to consider points in a sufficiently small neighborhood of the $(c^2/g, 0, 0)$ point of each frame.

The collection of frames, I_τ , as well as I_* and S , correspond exactly to the set of frames introduced in [1]. The only differences are, first, that now they are all local, in the sense that they are only well defined for regions in the neighborhood of their respective $(c^2/g, 0, 0)$ space points; and second, that now I_* and the I_τ frames are all considered to be freely falling toward W and the S frame is fixed with respect to W . Similarly to [1], the S frame may again be considered to be, relative to the viewpoint of I_* , a *Rindler noninertial frame*. The laboratory frame I_* we now call the *Einstein laboratory frame*, since now the “laboratory” is local and freely falling. We call the collection of inertial frames I_τ the *Boyer family of frames* as he was the first to introduce them in SED analysis of the Unruh-Davies effect [21].

The relativity principle as formulated by Einstein when proposing SR states that “all inertial frames are totally equivalent for the performance of all physical experiments.” [11] Before applying this principle to the freely-falling frames I_* and I_τ that we have defined above it is necessary to draw a distinction between these frames and inertial frames that are far away from any gravitating body, such as W . The free-fall trajectories, i.e. the geodesics, in the vicinity of any gravitating body, W , cannot be parallel over any arbitrary distance owing to the fact that W must be of finite size. This means that the principle of relativity can only be applied locally. This was precisely the limitation that Einstein had to put on his infinitely-extended Lorentz inertial frames of SR when starting to construct GR [11,20,22].

We adopt the principle of local Lorentz invariance (LLI) which can be stated, following Will [11], as “the outcome of a local nongravitational test experiment is independent of the velocity of the freely falling apparatus.” A non-gravitational test experiment is one for which self-gravitating effects can be neglected. We also adopt the assumption of space and time uniformity, which we call the uniformity assumption (UA) and which states that the laws of physics are the same at any time or place within the Universe. Again, following Will [11] this can be stated as “the outcome of any local nongravitational test experiment is independent of where and when in the universe it is performed.” We do not concern ourselves with physical or cosmological theories that in one way or another violate UA, e.g. because they involve spatial or temporal changes in fundamental constants [23].

Locally, the freely falling local Lorentz frames which we now designate with a subscript L — i.e. $I_{*,L}$ and $I_{\tau,L}$ — are entirely equivalent to the I_* and I_τ extended frames of [1]. The free-falling Lorentz frame $I_{\tau,L}$ locally is exactly the same as the extended I_τ . Invoking the LLI principle we can then immediately conclude that the electromagnetic zero-point field, or electromagnetic quantum vacuum, that can be associated with $I_{\tau,L}$ must be the same as that associated with I_τ . From the viewpoint of the local Lorentz frames $I_{*,L}$ and $I_{\tau,L}$ the body w is undergoing uniform acceleration and therefore for the same reasons as presented in [1] a peculiar acceleration-dependent force arises, that for concreteness we shall call a drag force.

These formal arguments demonstrate that the analyses of [1] which found the existence of a Rindler frame force in an accelerating reference frame translate and correspond exactly to a reference frame fixed above a gravitating body. In the same manner that light rays are deviated from straight-line propagation by

a massive gravitating body W , the other forms of electromagnetic radiation, including the electromagnetic zero-point field rays (in the SED approximation), are also deviated from straight-line propagation. Not surprisingly this creates an anisotropy in the otherwise isotropic electromagnetic quantum vacuum.

In [1] we interpreted the drag force exerted by the electromagnetic quantum vacuum radiation as the inertia reaction force of an object that is being forced to accelerate through the electromagnetic quantum vacuum field. Accordingly, in the present situation, the associated nonrelativistic form of the inertia reaction force should be

$$\mathbf{f}_*^{zp} = -m_i \mathbf{g}_w \quad (22)$$

where \mathbf{g}_w is the acceleration with which w appears in the local inertial frame I_* . As shown in [1] the coefficient m_i is

$$m_i = \left[\frac{V_0}{c^2} \int \eta(\omega) \frac{\hbar \omega^3}{2\pi^2 c^3} d\omega \right] \quad (23)$$

where V_0 is the proper volume of the object, c is the speed of light, \hbar is Planck's constant divided by 2π and $\eta(\omega)$, where $0 \leq \eta(\omega) \leq 1$, is a function that spectralwise represents the relative strength of the interaction between the zero-point field and the massive object, interaction which acts to oppose the acceleration. If the object is just a single particle, the spectral profile of $\eta(\omega)$ will characterize the electromagnetically-interacting particle. It can also characterize a much more extended object, i.e. a macroscopic object, but then the $\eta(\omega)$ will have much more structure (in frequency). We should expect different shapes for the electron, a given quark, a composite particle like the proton, a molecule, a homogeneous dust grain or a homogeneous macroscopic body. In the last case the $\eta(\omega)$ becomes a complicated spectral opacity function that must extend to extremely high frequencies such as those characterizing the Compton frequency of the electron and even beyond.

Now, however, what appears as inertial mass, m_i , to the observer in the local $I_{*,L}$ frame is of course what corresponds to passive gravitational mass, m_g , and it must therefore be the case that

$$m_g = \left[\frac{V_0}{c^2} \int \eta(\omega) \frac{\hbar \omega^3}{2\pi^2 c^3} d\omega \right]. \quad (24)$$

As done in [1], Appendix B, it can be shown that the right hand side indeed represents the energy of the electromagnetic quantum vacuum enclosed within the object's volume and able to interact with the object as manifested by the $\eta(\omega)$ coupling function. A more thorough, fully covariant development can also be implemented to show that the force expression of eqn. (1) can be extended to the relativistic form of the inertia reaction force as in [1], Appendix D. (This development also served to obtain the final form of m_i given above in eqn. (3) eliminating a spurious $4/3$ factor.)² Summarizing what we have shown in this section is that if a force \mathbf{f} is applied to the w body just large enough to prevent it from falling toward the body W , then in the non-relativistic case that force is given by

$$\mathbf{f} = -m\mathbf{g} \quad (25)$$

where we have dropped the nonessential subscripts i and g and superscripts, because it is now clear that $m_i = m_g = m$ follows from the quantum vacuum inertia hypothesis.

The physical basis for the principle of equivalence is the fact that accelerating through the electromagnetic quantum vacuum is identical to remaining fixed in a gravitational field and having the electromagnetic quantum vacuum fall past on curved geodesics. In both situations the observer will experience an asymmetry

² We use this opportunity to correct a minor transcription error that appeared in the printed version of Ref. [1]. In Appendix D, p. 1100, the minus sign in eqn. (D8) is wrong. It should read

$$\eta^\nu \eta_\nu = 1 \quad (D8)$$

and the corresponding signature signs in the line just above eqn. (D8) are the opposite of what was written and should instead read $(+ - - -)$.

in the radiation pattern of the electromagnetic quantum vacuum which results in a force — either the inertia reaction force or weight — which become then within this more general Einsteinian perspective the same thing.

5. CONSISTENCY WITH EINSTEIN'S GENERAL RELATIVITY

The statement that $m_i = m_g$ constitutes the weak equivalence principle (WEP). Its origin goes back to Galileo and Newton, but it now appears, as shown in the previous section, that this principle is a natural consequence of the quantum vacuum inertia hypothesis. The strong equivalence principle (SEP) of Einstein consists of the WEP together with LLI and the UA. Since the quantum vacuum inertia hypothesis and its extension to gravity allow us to obtain the WEP assuming LLI and the UA, this approach is consistent with all theories that are derived from the SEP. In addition to GR, the Brans-Dicke theory is derived from SEP as are other lesser known theories [11], all distinguished from each other by various particular assumptions.

All theories that assume the SEP are called *metric theories*. They are characterized by the fact that they contemplate a bending of spacetime associated with the presence of matter. Two important consequences of the LLI-WEP-UA combination are that light bends in the presence of matter and that there is a gravitational Doppler shift. Since the quantum vacuum inertia hypothesis is consistent with this same combination, it would also require that light bends in the presence of gravitational fields. This can, of course, be interpreted as a change in spacetime geometry, the standard interpretation of GR. However this does not mean that we have explained the mechanism for the actual bending of space-time in the vicinity of a material object. This is the origin of so-called active gravitational mass that still requires an explanation within the viewpoint of the quantum vacuum inertia hypothesis.

6. A RECENT ALTERNATIVE VACUUM APPROACH TO GRAVITY

The idea that the vacuum is ultimately responsible for gravitation is not new. It goes back to a proposal of Sakharov [24] based on the work of Zeldovich [25] in which a connection is drawn between Hilbert-Einstein action and the quantum vacuum. This leads to a view of gravity as “a metric elasticity of space” (see Misner, Thorne and Wheeler [26] for a succinct review of this concept). Following Sakharov's idea and using the techniques of SED, Puthoff proposed that gravity could be construed as a form of van der Waals force [27]. Although interesting and stimulating in some respects, Puthoff's attempt to derive a Newtonian inverse square force of gravity proves to be unsuccessful [28,29,30,31].

An alternative approach has recently been developed by Puthoff [32] that is based on earlier work of Dicke [33] and of Wilson [34]: a polarizable vacuum model of gravitation. In this representation, gravitation comes from an effect by massive bodies on both the permittivity, ϵ_0 , and the permeability, μ_0 , of the vacuum and thus on the velocity of light in the presence of matter.

That is clearly an alternative theory to GR since it does not involve curvature of spacetime. On the other hand, since spacetime curvature is by definition inferred from light propagation in relativity theory, the polarizable vacuum gravitation model may be labelled a pseudo-metric theory of gravitation since the effect of variation in the dielectric properties of the vacuum by massive objects on light propagation are approximately equivalent to GR spacetime curvature as long as the fields are sufficiently weak. In the weak-field limit, the polarizable vacuum model of gravitation duplicates the results of GR, including the classic tests (gravitational redshift, bending of light near the Sun, advance of the perihelion of Mercury). Differences appear in the strong-field regime, which should lead to interesting tests.

7. DERIVATION OF NEWTON'S LAW OF GRAVITATION

Our approach allows us to derive a Newtonian form of gravitation in the weak-field limit based on the quantum vacuum inertia hypothesis, local Lorentz invariance and geometrical considerations. In [1] we showed how an asymmetry that appears in the radiation pattern of the electromagnetic quantum vacuum, when viewed from an accelerating reference frame, leads to the appearance of an apparent non-zero momentum flux. Individual and collective interaction between the electromagnetically-interacting particles (quarks and electrons) comprising a material object and the electromagnetic quantum vacuum generates a reaction force that may be identified with the electromagnetic contribution to the inertia reaction force as it has the

right form for all velocity regimes. In particular in the low velocity limit it is exactly proportional to the acceleration of the object.

In §4 we showed from formal arguments based on the principle of local Lorentz invariance (LLI) that an exactly equivalent force must originate when an object is effectively accelerated with respect to free-falling Lorentz frames by virtue of being held fixed above a gravitating body, W . We infer that the presence of a gravitating body must distort the electromagnetic quantum vacuum in exactly the same way at any given point as would the process of acceleration such that $\mathbf{a} = -\mathbf{g}$. Simple geometrical arguments [35] now suffice to show that the gravitational force can only be the Newton inverse-square law with distance (in the weak field limit).

It has been shown above that (outside W) the \mathbf{g} field of eqn. (25) generated by W is central, i.e., centrally distributed with spherical symmetry around W . It has to be radial, with its vectorial direction parallel to the corresponding radius vector \mathbf{r} originating at the center of mass of W where we locate the origin of coordinates. The spherical symmetry implies that \mathbf{g} is radial in the direction $-\hat{\mathbf{r}}$, and depends only on the r -coordinate, $\mathbf{g} = -\hat{\mathbf{r}}g(r)$.

The field is clearly generated by mass. A simple symmetry argument based on Newton's Third Law, here omitted for brevity, shows that indeed if we scale the mass M of W by a factor α to become αM then the resulting \mathbf{g} must be of the form $\alpha\mathbf{g}$. If M goes to zero, \mathbf{g} disappears. The mass M must be the source of field lines of \mathbf{g} , and these field lines can be discontinuous only where mass is present. The field lines can be neither generated nor destroyed in free space. Since \mathbf{g} is the force on a unit mass, we must expect that \mathbf{g} behaves as a vector, and specifically that \mathbf{g} follows the laws of vector addition. Namely, if two masses M_1 and M_2 in the vicinity of each other generate fields \mathbf{g}_1 and \mathbf{g}_2 respectively, the resulting \mathbf{g} at any given point in space should be the linear superposition of the \mathbf{g}_1 and \mathbf{g}_2 fields at that given point, namely their vector sum

$$\mathbf{g} = \mathbf{g}_1 + \mathbf{g}_2. \quad (26)$$

Finally, from the argument that leads to eqn. (25) we can see that the field \mathbf{g} must be unbounded, extending essentially to infinity.

With all of these considerations, clearly the lines of \mathbf{g} must obey the continuity property outside W . If there is no mass present inside a volume, V , enclosed by a surface S , we expect, using Gauss' divergence theorem, that

$$0 = \oint_{S(V)} \mathbf{g} \cdot \mathbf{n} \, dS = \int_V \nabla \cdot \mathbf{g} \, dV, \quad (27)$$

but since V is arbitrary this tells us that outside the massive body W

$$\nabla \cdot \mathbf{g} = 0, \quad r > R. \quad (28)$$

In the presence of our single, spherically symmetric massive body, W , but outside that body, \mathbf{g} is radial with respect to the center of W . Hence we must have $\mathbf{g} = \hat{\mathbf{r}}\Phi(r)$, and therefore from eqn. (28), because Φ is only a function of r , it necessarily follows that \mathbf{g} is of the form

$$\mathbf{g} \sim -\hat{\mathbf{r}}\frac{1}{r^2}, \quad (29)$$

where the restriction $r > R$ is hereafter understood. Since the field \mathbf{g} is also proportional to the mass M which is its origin, we conclude that \mathbf{g} must be of the form

$$\mathbf{g} = -\hat{\mathbf{r}}\frac{GM}{r^2}, \quad (30)$$

where G is a proportionality constant and from eqn. (17) we have that

$$\mathbf{F} = -\hat{\mathbf{r}}\frac{GMm}{r^2}, \quad (31)$$

which is Newton's law of gravitation. It is remarkable that after finding the central and radial character of \mathbf{g} by means of the vacuum approach of our quantum vacuum inertia hypothesis, one can immediately

obtain Newtonian gravitation, an endeavour keenly but unsuccessfully pursued for quite some time from the viewpoint of the vacuum fields [24,25,26] and in particular of SED [27,28,29,30,31]. In this last case (SED), it was proposed that gravity was a force of the van der Waals form, a view which has been shown to be unsuccessful [31].

As a final point here we observe that when the two bodies of masses M and m are reduced to point particles, a simple argument based on the symmetry of the situation shows that the two roles of the masses m and M (passive and active) can be interchanged and that similarly the force on M due to m should be equal in magnitude but opposite in direction to that of eqn. (22) so that Newton's third law of mechanics is satisfied.

8. ON THE ORIGIN OF WEIGHT

We have established that the quantum vacuum inertia hypothesis leads to a force in eqn. (22) which adequately explains the origin of weight in a Newtonian view of gravitation. How is this consistent with the geometrodynamics view of GR? Geometrodynamics specifies the effect of matter and energy on an assumed pliable spacetime metric. That defines the geodesics which light rays and freely-falling objects will follow. However there is nothing in geometrodynamics that points to the origin of the inertia reaction force when geodesic motion is prevented, which manifests in special circumstances as weight.

Geometrodynamics merely assumes that deviation from geodesic motion results in inertial forces. That is, in fact, true, but as it stands it is devoid of any physical insight. What we have shown above is that an identical asymmetry in the quantum vacuum radiation pattern will arise due to either true acceleration or to effects on light propagation by the presence of gravitating matter. In the case of true acceleration, the resulting force is the inertia reaction force. In the case of being held stationary in a non-Minkowski metric, the resulting force is the weight, which is also the enforcer of geodesic motion for freely-falling objects.

9. A PHANTOM ANOMALY

Following the reasoning of §7 one would conclude that at every point of fixed r above W there is an inflowing energy-momentum flux from the falling electromagnetic quantum vacuum. This would seem to imply a continuous energy flux toward and into W , an apparently paradoxical situation consisting of quantum vacuum-originating energy streaming toward W from all directions in space. But this paradox is only apparent, as we show.

It is true that for a freely falling observer attached to a local inertial frame $I_{\tau,L}$ which is falling past the surface of the Earth and towards its center (§4), the object w , that we take now to be located on the surface of the Earth, appears to be uniformly accelerated with constant proper acceleration $\mathbf{g}_w = -\mathbf{g}$. Thus this observer concludes as he freely falls that the mass of w must grow as $\gamma_\tau m$, where m is the rest mass of w and γ_τ is the Lorentz factor, $(1 - \beta_\tau^2)^{-1/2}$.

Concomitantly, this observer of $I_{\tau,L}$, in light of our discussion in App. B of Ref. [1] and because of the SEP, asserts that the energy content of w , which for simplicity we take here as purely electromagnetic and coming from the electromagnetic vacuum, is steadily increasing. Eqn. (33) of Ref. [1] gives an energy growth rate $dE/dt = \mathbf{f} \cdot \mathbf{v}$, where \mathbf{f} in the present case corresponds to the force of support against the Earth's gravity exerted on w by the surface of the Earth. Clearly then this freely falling observer of I_τ concludes that objects on the surface of the Earth are steadily increasing their internal energy at the expense of the falling vacuum, namely of the electromagnetic zero-point field of I_τ , that freely falls with I_τ .

However when exactly the same consideration is made but now from the point of view of an observer fixed on the surface of the Earth and in the neighborhood of w , the conclusion is quite different. In this case the observer is attached to S , the local Rindler non-inertial frame where w is fixed and that coincides with the surface of the Earth around the spot where w is located. To calculate dE/dt as seen from S we use the local instantaneously comoving, freely-falling inertial frame $I_{*,L} = I_{\tau=0,L}$ which at that point in time co-moves with w . In that frame $I_{*,L}$ the particle velocity is of course zero, and therefore the rate of energy growth is $dE/dt = 0$. So the observer in S concludes that w does not change its energy content.

The above shows that for an observer on the surface of the Earth there is no real transfer of energy from the vacuum to objects on the surface of the Earth. In this respect the energy growth above, that a freely falling observer sees as time progresses, is purely a kinematical effect that comes from the fact that in

the freely-falling frame the velocity of the object w is growing with time and does not vanish except at the instant of coincidence of $I_{*,L}$ with S , at $\tau = 0$. In special relativity, a constantly accelerating spaceship is said by an external observer to be increasing its mass steadily as $\gamma(\tau)m$, but of course an observer onboard that craft will not perceive any relativistic mass increase.

On the other hand as seen in S , for the frame at the surface of the Earth, even though dE/dt vanishes, the force \mathbf{f} does not vanish, as shown in Ref. [1]. The force \mathbf{f} is needed to compensate the force \mathbf{f}_{zp} , i.e. $\mathbf{f}_{zp} = -\mathbf{f}$, exerted on w by the freely-falling vacuum of I_τ , that indeed corresponds to what we called above the Rindler frame force.

10. DISCUSSION

In light of what has been proposed herein, what can we elucidate about the nature of the gravitational field? In the low fields and low velocities version, or the Newtonian limit, we have seen above that gravity manifests itself as the attractive force per unit mass, \mathbf{g} , of eqn. (30) that pulls any massive test body present at a given point in space towards the body W that originates the field. Since we assumed the Einstein LLI principle and from this derived the WEP, this, together with the very natural UA of invariance in the laws of physics throughout universal spacetime, leads us to the Einstein SEP which necessarily implies the spacetime bending representation of the generalized gravitational field [11,20].

A simple thought experiment (Einstein's lift) immediately shows that light rays propagate along geodesics, and more specifically along null geodesics [20,26,36]. The spacetime bending is dramatically evident when a light ray goes from one side to the other of the freely falling elevator. For the observer attached to the elevator's frame that indeed acts as a LLI frame, the light ray propagates in a straight line from one side to the other of the elevator. But for the stationary observer who sees the elevator falling with acceleration \mathbf{g} , the light ray bends along a path that locally is seen as a parabolic curve. Undoubtedly the most natural explanation for the stationary observer is that spacetime bends and therefore the association that this bending is a manifestation of the gravitational field of W , or rather that this bending of spacetime is the gravitational field itself [36].

Starting from the above fact taken as a given, the various metric theories proceed from there to formulate their equations. In the version of Brans and Dicke a scalar-tensor field is assumed. In Einstein's GR only a tensor field is proposed. Following this maximally simplistic proposal and guided by general considerations of general covariance (the need of arbitrary coordinates and tensor laws) Einstein was led to his field equations in the presence of matter... and then GR naturally unfolded [11,20,26,36].

We have shown here that our inertia proposal of [1] leads us, when limited by the LLI principle, to the metric theories and therefore that it is consistent with those theories and in particular with Einstein's GR. In addition, there is the following interesting feature of our proposal.

From our analysis in [1], and in particular in Appendix B of [1], it was made clear that within the quantum vacuum inertia hypothesis proposed therein, the mass of the object, m , could be viewed as the energy in the equivalent electromagnetic quantum vacuum field captured within the structure of the object and that readily interacts with the object. This view properly and accurately matched with the complementary view, exposed in other parts of the paper [1], that presented inertia as the result of a vacuum reaction effect, a kind of drag force exerted by the vacuum field on accelerated objects. Quantitatively, both approaches lead to exactly the same inertial mass and moreover they are partly complementary. Both viewpoints are needed. One could not exist without the other. They were the two sides of the same coin.

The question is now why massive objects, when freely falling, also follow geodesic paths. The tempting view suggested here is that, as massive bodies have a mass that is made of the vacuum electromagnetic energy contained within their structure and that readily interacts with such structure (according to our analysis of [1], Appendix B), it is no surprise that geodesics are their natural path of motion during free fall. Electromagnetic radiation has been shown by Einstein to follow precisely geodesic paths. The only difference now is that, as the radiation stays within the accelerated body structure and is contained within that structure and thereby its energy center moves subrelativistically, these geodesics are just time-like and not null ones as in the case of freely propagating light rays.

We illustrate this with an example. Imagine a freely-falling electromagnetic cavity with perfectly reflecting walls of negligible weight, so that all the weight is due to the enclosed radiation. A simple plane wave mode decomposition shows that although individual wavetrains do still move at the speed of light, the

center of energy of the radiation inside the cavity moves subrelativistically as the wavetrains reflect back and forth. The wavetrains do indeed propagate along null geodesics, but the center of energy propagates only along a time-like geodesic.

Neither our approach nor the conventional presentations of GR for that matter, can offer a physical explanation of the mechanism of the bending of spacetime as related to energy density. Misner, Thorne and Wheeler [26] present six different proposed explanations. The sixth is the one we already mentioned due to Sakharov [24,25] which starts from general vacuum considerations. As our approach starts also from vacuum considerations, it naturally fits better the concept of the conjecture of Sakharov [24] and Zeldovich [25] than the other proposals but it is not inconsistent with any of them. In particular the strictly formal proposal of Hilbert [26] that introduces the so-called Einstein-Hilbert Action, is also at the origin of the Sakharov proposal. We plan to devote more work to exploring the connection of our inertia [1] and gravity approach to the approach proposed by Sakharov [24]. This we leave for a future publication.

11. CONCLUSIONS

The principal conclusions of this paper are:

(1) *Identity of inertial mass with gravitational mass, $m_i = m_g$ naturally follows from the quantum vacuum inertia hypothesis.* It has been shown that the approach of [1] contains this peculiar feature that, so far and as we know, has never been explained. Rindler [20] calls this feature “a very mysterious fact,” as indeed it has been up to the present. We expect with this work to have shed some light on this peculiar feature.

(2) *The quantum vacuum inertia hypothesis is consistent with Einstein’s GR.* We have already commented above, in particular in §8, on this interesting feature presented in §5 that puts the vacuum inertia approach of [1] within the mainstream thought of contemporary gravitational theories, specifically within that of theories of the metric type and in particular in agreement with GR.

(3) *Newton’s gravitational law naturally follows from the quantum vacuum inertia hypothesis.* By means of a simple argument based on potential theory we show how to obtain in a natural way Newton’s inverse square force with distance from our vacuum approach to inertia of [1]. The simplicity of our approach contrasts with previous attempts to accomplish this within the framework of SED theory.

(4) *An origin of weight and a physical mechanism to enforce motion along geodesic trajectories for freely-falling objects is contained in the hypothesis.* We have shown how this approach to inertia answers a fundamental question left open within GR, viz. is there a physical mechanism that generates the inertia reaction force when non-geodesic motion is imposed on an object and which can manifest specifically as weight. Or put another way, while geometrodynamics dictates the spacetime metric and thus specifies geodesics, is there an identifiable mechanism for enforcing motion along geodesic trajectories? The quantum vacuum inertia hypothesis represents a significant first step in providing such a mechanism.

(5) *A physical basis is provided for the relativistic mass increase.* In the approach of the quantum vacuum inertia hypothesis, it becomes physically evident how the Lorentz factor γ_τ , which characterizes a space-time geometry relationship, acquires the physical role of relativistic mass increase parameter. It also becomes clear why the relativistic mass increase must become infinite at c : One cannot propagate the effect of any forces originating in zero-point field effects at speeds faster than c .

(6) *The interior (excluding the walls) of a cavity resonator is presented as the archetype of a system wherein only the electromagnetic vacuum contributes to the mass.* This allows an expression for a resulting inertial mass with no free parameters. This mass in principle can be measured.

(7) *An experimental prediction has been made that the mass of the resonant electromagnetic zero-point field modes within a cavity should add to the mass of the cavity structure.*

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